ON THE DETERMINATION OF A SERIES OF STURM'S FUNCTIONS BY THE CALCULATION OF A SINGLE DETERMINANT.*

By E. B. Van Vleck.

1. In the most familiar form of Sturm's theorem, the number of roots of a real polynomial

$$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$$

included between two real limits, is determined with the aid of a series of functions consisting of the polynomial, its derivative, and the successive remainders, taken with appropriate signs, which occur in the process of finding the greatest common divisor of the polynomial and its derivative. In place of the derivative any other polynomial

$$f_1(x)=b_0x^n+b_1x^{n-1}+\cdots\cdot+b_n$$

may be employed as the divisor, if it has an odd number of roots between every two consecutive roots of $f(x)$. The resulting Sturm remainders, modified at pleasure by multiplication with a positive numerical factor, will here be denoted by $f_2, f_3, \ldots\ldots f_{n+1}$. The calculation of these remainders by the process of division is, in general, exceedingly laborious. The object of this paper is to show that the determination of the entire series of functions, with their proper signs, can be reduced to the systematic computation of a single determinant. The method suggested, it is believed, is well adapted to practical computation and, except in extremely simple cases, greatly abridges the labor.

To make clear the relation of the present article to previous work by others we shall bring together in §2 the various cases in which the coefficients of a series of Sturm's functions have all been expressed as minors of a common determinant.† We shall then go on to show, in §3, that a new case can be obtained by supplementing or completing certain results given incidentally in one of Sylvester's papers. Upon this new case will be based our method of computing the remainders.

---

* This paper was read before the American Mathematical Society at the meeting of Feb. 25, 1899.

† These cases lie scattered in the mathematical literature, and do not seem to have been heretofore brought together.

2. The expression of the coefficients of a series of Sturm's remainders as minors of a common determinant was first effected by Jacobi* in 1835. Eighteen years later, in ignorance of Jacobi's work, the same result was also obtained by Sylvester.† Both writers express the coefficients as minors of Bezout's form of the resultant of f and f_1, namely

$$\begin{vmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ B_{21} & B_{22} & \dots & B_{2n} \\ \dots & \dots & \dots & \dots \\ B_{n1} & B_{n2} & \dots & B_{nn} \end{vmatrix}, \quad B_{ij} = B_{ji}$$

in which the constituents are certain well-known quadratic functions of the coefficients of f and f_1. The $n - i + 1$ coefficients of the ith remainder are the minors obtained from the first i rows by associating those constituents which are found in the first $i - 1$ columns with those of each succeeding column in turn.

The second series of Sturm's functions whose coefficients have been expressed in terms of a common determinant has a somewhat different composition.‡ If $f_1(x)$ is taken to be a polynomial of $n-1$th degree, the quotient f_1/f can be expanded, on the one hand, into a continued fraction

$$\frac{1}{a_1 x + b_1} - \frac{1}{a_2 x + b_2} - \frac{1}{a_3 x + b_3} - \dots - \frac{1}{a_n x + b_n};$$

on the other hand, into an infinite series

$$\frac{c_0}{x} + \frac{c_1}{x^2} + \frac{c_2}{x^3} + \dots$$

The denominators of the convergents of the continued fraction, inclusive of f itself, constitute a Sturm's series under the same restrictions for f_1 as in the case of Sturm's remainders. When obtained directly from the infinite series, the denominators have the form

$$\begin{vmatrix} c_0 & c_1 & c_2 & \dots & c_{i-1} & 1 \\ c_1 & c_2 & c_3 & \dots & c_i & x \\ c_2 & c_3 & c_4 & \dots & c_{i+1} & x^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_i & c_{i+1} & \dots & \dots & c_{2i-1} & x^i \end{vmatrix},$$

* *Crelle*, Vol. 15.

† *Philosophical Transactions*, 1853. See, in particular, Sylvester's remarks on p. 483.

‡ Netto's *Algebra*, Siebente Vorlesung.

and their coefficients are therefore minors of the determinant

$$\begin{vmatrix} c_0 & c_1 & c_2 & \dots\dots & c_n \\ c_1 & c_2 & c_3 & \dots\dots & c_{n+1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ c_n & c_{n+1} & \dots & \dots & c_{2n} \end{vmatrix}$$

In the special case in which f_1 is placed equal to the derivative of f, the constituent c_i is the sum of the ith powers of the roots of $f(x) = 0$, and the determinant, omitting the last row and column, becomes a familiar form of the discriminant of this equation.

The coefficients of the denominators of the same convergents have also been expressed* as minors of Sylvester's form of the resultant of f and f_1,

$$R = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & . & . & a_n & 0 & 0 & . & . & 0 \\ b_0 & b_1 & b_2 & b_3 & . & . & b_n & 0 & 0 & . & . & 0 \\ 0 & a_0 & a_1 & a_2 & . & . & a_{n-1} & a_n & 0 & . & . & 0 \\ 0 & b_0 & b_1 & b_2 & . & . & b_{n-1} & b_n & 0 & . & . & 0 \\ 0 & 0 & a_0 & a_1 & . & . & a_{n-2} & a_{n-1} & a_n & . & . & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & . & b_0 & b_1 & b_2 & b_3 & . & . & b_n \end{vmatrix} \qquad (1)$$

thus making a third case in which the coefficients of a Sturm's series have been expressed as minors of a common determinant. For the computation of the functions all these cases seem to be open to the common objection that the independent computation of the minors is far more laborious than the derivation of Sturm's remainders by the usual process of division. We shall therefore seek a series of functions whose coefficients form such a set of minors as permit of calculation simultaneously with the determinant.

3. To this end consider the set of polynomials which Sylvester† obtained by so applying his dialytic method of elimination — commonly used to obtain the above form of R — as to eliminate only the first i highest powers of x between $f(x) = 0$ and $f_1(x) = 0$, i taking successively the values 1, 2, , ... n. The coefficients of the ith polynomial thus obtained are the minors formed from the first $2i$ rows of (1) by associating those constituents which are contained in the first $2i - 1$ columns with those of each

* Netto's *Algebra*, *l. c.*

† *Philosophical Transactions*, 1853, p. 426–7.

succeeding column in turn. Thus, for example, the second of these polynomials is

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & b_0 & b_1 & b_2 \end{vmatrix} x^{n-2} + \begin{vmatrix} a_0 & a_1 & a_2 & a_4 \\ b_0 & b_1 & b_2 & b_4 \\ 0 & a_0 & a_1 & a_3 \\ 0 & b_0 & b_1 & b_3 \end{vmatrix} x^{n-3} + \cdots + \begin{vmatrix} a_0 & a_1 & a_2 & 0 \\ b_0 & b_1 & b_2 & 0 \\ 0 & a_0 & a_1 & a_n \\ 0 & b_0 & b_1 & b_n \end{vmatrix} = 0.$$

Sylvester has shown * that these polynomials are what he terms the "simplified residues" which occur in the process of finding the greatest common divisor of f and f_1, — that is, each remainder is so expressed that its coefficients are rational functions of the a and the b without a common divisor. In consequence each polynomial differs from the corresponding (simplified) Sturm remainder only by a numerical factor independent of the a and b. To demonstrate that these polynomials are *identical* with Sturm's remainders, it is left for us to prove the vital point that they agree in sign. It will evidently suffice to compare the leading coefficients of our polynomials with the corresponding coefficients of Sturm's remainders.

We will assume at the outset of the proof that the signs of a_0 and b_0 have been so taken as to be positive. This insures that the leading coefficient $\begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix}$ of our first polynomial shall agree in sign with the corresponding coefficient of the first of Sturm's remainders. Beginning with this polynomial and remainder, the degree of each succeeding polynomial, respectively remainder is, in general, one less than that of the preceding. Now Sylvester has remarked that when this is true of Sturm's remainders, and the leading coefficient of any remainder vanishes, the leading coefficients of the preceding and following remainders have opposite signs. This is evident; for let $L_0x^r + L_1x^{r-1} + L_2x^{r-2}$ and $M_0x^{r-1} + M_1x^{r-2} + M_2x^{r-3}$ denote the first three terms of two consecutive Sturm remainders. By division the first coefficient of the next simplified remainder is found to be

$$M_0(L_0 M_2 - M_0 L_2) - M_1(L_0 M_1 - M_0 L_1)$$

which for $M_0 = 0$ has a sign opposite to that of L_0. We will now prove that the same property also holds for our polynomials. It follows then at once that their leading coefficients agree in sign with the corresponding coefficients of Sturm's remainders.

---

* See the last reference. The same fact can also be inferred from an article by Netto in the *Festschrift der mathematischen Gesellschaft in Hamburg*, 1890, but the line of proof is not so simple or fundamental as Sylvester's.

Consider first the case in which $\begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix}$, the leading coefficient of the first polynomial, vanishes. The leading coefficient of the next polynomial is then equal to $-\begin{vmatrix} a_0 & a_2 \\ b_0 & b_2 \end{vmatrix}^2$ and consequently differs in sign from the first coefficient of $f_1(x)$.

Take next the leading coefficients of any three consecutive polynomials,

$$D_{2i-2} = \begin{vmatrix} a_0 & a_1 & a_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_0 & b_1 & b_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & a_0 & a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot\,\cdot & b_0 & b_1 & b_2 & \cdot\,\cdot & b_i \\ 0 & 0 & 0 & \cdot\,\cdot & 0 & a_0 & a_1 & \cdot\,\cdot & a_{i-1} \\ 0 & 0 & 0 & \cdot\,\cdot & 0 & b_0 & b_1 & \cdot\,\cdot & b_{i-1} \end{vmatrix}, \quad D_{2i} = \begin{vmatrix} a_0 & a_1 & a_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_0 & b_1 & b_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & a_0 & a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot\,\cdot\,\cdot & b_0 & b_1 & b_2 & \cdot\,\cdot\,\cdot & b_{i+1} \\ 0 & 0 & 0 & \cdot\,\cdot\,\cdot & 0 & a_0 & a_1 & \cdot\,\cdot\,\cdot & a_i \\ 0 & 0 & 0 & \cdot\,\cdot\,\cdot & 0 & b_0 & b_1 & \cdot\,\cdot\,\cdot & b_i \end{vmatrix},$$

$$D_{2i+2} = \begin{vmatrix} a_0 & a_1 & a_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_0 & b_1 & b_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & a_0 & a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot\,\cdot\,\cdot & b_0 & b_1 & b_2 & \cdot\,\cdot\,\cdot & b_{i+2} \\ 0 & 0 & 0 & \cdot\,\cdot\,\cdot & 0 & a_0 & a_1 & \cdot\,\cdot\,\cdot & a_{i+1} \\ 0 & 0 & 0 & \cdot\,\cdot\,\cdot & 0 & b_0 & b_1 & \cdot\,\cdot\,\cdot & b_{i+1} \end{vmatrix}$$

and multiply the first by the last with the aid of the following theorem of determinants:

"The product of a determinant and any one of its minors M is expressible as an aggregate of products of pairs of minors: the first factors of the products being obtained by taking q rows in which the rows of M are included and forming from them every minor of the qth order which contains M; the second factor of any product being that minor which includes M and the complementary of the first factor; and the sign of any product being fixed by transforming the second factor so as to have its principal diagonal coincident with those of the two minors which it was formed to include, and then taking + or − according as the sum of the numbers indicating the rows and columns from which the first factor was formed is even or odd." [Muir's *Theory of Determinants*, p. 128, § 90.]

In applying this proposition, the first of our three determinants can be taken out as a minor from the third in several ways. We will consider it to

be the minor which is obtained by omitting the first two and the last two rows, also the first column and the last three columns. We will choose for our q rows all the rows from the third to the last inclusive. Of the determinants of the $2i$th order which can be formed from these rows so as to contain D_{2i} as a minor, all but three include the first column which contains only zero constituents, and hence vanish identically. The remaining three are the determinants obtained by omitting, in addition to the first column, either the last column, the next to the last, or the second preceding the last. Of these three determinants the first is D_{2i}, while the third in accordance with the above theorem, is to be multiplied by D_{2i}. Hence when $D_{2i}=0$, the only partial product to be considered is the remaining determinant multiplied into a factor which is easily seen to be its negative. It follows that $D_{2i-2}\,D_{2i+2}$ is then negative. In other words, whenever the leading coefficient of any one of our polynomials vanishes, the leading coefficients of the preceding and following polynomials have opposite signs, as was to be proved. Since the same property holds for the remainders, and since, also, the first two polynomials and remainders have been shown to be identical, the subsequent polynomials must also be identical with the corresponding remainders.

The correctness of the signs of the leading coefficients of our polynomials might also have been inferred by reducing the resultant R to Bezout's form after the manner given in *Baltzer's Determinantentheorie*, § 11, section 14. Each principal minor* D_{2i} is at the same time transformed into the principal minor of Bezout's determinant which is of the ith order, that is, into the leading coefficient of one of the remainders.

4. We have thus demonstrated that f, f_1, *and the successive polynomials which are formed from the first $2i$ rows of R, $(i=1, 2, \ldots n)$ constitute a Sturm series.* A comparison of this series with the two series previously cited, whose coefficients were expressed as minors of a common determinant, shows certain advantages from a theoretical standpoint for either of the latter two. The order of the fundamental determinant is only one-half as great, and the determinant has the further advantage of being symmetrical with respect to the principal diagonal. But these advantages are largely offset by the fact that the constituents of the determinant (1) require no preliminary calculation. For example, if the number of real roots of $f(x)=0$ be desired, we need only to write down the series of principal minors of even order†

*German, *Hauptunterdeterminante*. By the term "principal minor" of the ith order is here to be understood the minor which contains the constituents common to the first i rows and columns.

†This series of determinants has been previously set up by Hurwitz, see *Math. Ann.*, Vol. 46.

$$a_0,\ b_0,\ \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix},\ \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & b_0 & b_1 & b_2 \end{vmatrix},\ \ldots R, \tag{2}$$

these being the coefficients of the first terms, upon which the signs of the successive functions depend when we place $x = \pm \infty$. *The number of changes of sign in (2) gives the number of pairs of imaginary roots of the polynomial.* To determine how many of the real roots are positive, we must also take into consideration the constant terms of the polynomials, for which we have the expressions

$$a_n,\ b_n,\ \begin{vmatrix} a_0 & a_n \\ b_0 & b_n \end{vmatrix},\ \begin{vmatrix} a_0 & a_1 & a_2 & 0 \\ b_0 & b_1 & b_2 & 0 \\ 0 & a_0 & a_1 & a_n \\ 0 & b_0 & b_1 & b_n \end{vmatrix},\ \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ b_0 & b_1 & b_2 & b_3 & b_4 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_n \\ 0 & 0 & b_0 & b_1 & b_2 & b_n \end{vmatrix},\ \ldots R. \tag{3}$$

The excess of the number of changes of sign in (2) over the number of changes of sign in (3) is equal to the required number of positive roots. The series (2) has also a significance when $f_1(x)$ is not subject to the restriction stated in §1, giving then the value of Cauchy's index for the quotient f_1/f.

5. As soon as we leave abstract theory and pass to calculation, our series of functions has a most decided advantage, in that the significant constituents of any two consecutive rows of the fundamental determinant (1) are the same as the constituents of the two preceding rows. If therefore in any row the values of the constituents are changed by the addition of a multiple of the preceding row, exactly the same change can be made in the constituents of each alternate row thereafter without altering the value of any minor which appears as a coefficient in one of our Sturm's functions. Moreover the alternate rows can each be multiplied by the same positive number without affecting either the form of the determinant or the signs of these minors. If then, in this manner, beginning with the second row of the determinant and proceeding downward, we reduce to zero all constituents which lie to the left of the principal diagonal, the repetitive character of the determinant will always be preserved below the last row in which this reduction has been effected, while at the same time the number of zero constituents rapidly increases. At each stage of the reduction it will therefore be necessary to retain only two rows, the one in which the reduction is about to be made and the row above it with the aid of which the

reduction is effected. As soon as the first constituent of the former row has been reduced to zero, the latter row, with its constituents moved one place to the right, is to be written down below it as the next row to be modified, and so on. The determinant can thus rapidly be brought to the form

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \ldots a_n & 0 & 0 \ldots 0 \\ 0 & C_{20} & C_{21} & C_{22} \ldots C_{2,n-1} & 0 & 0 \ldots 0 \\ 0 & 0 & C_{30} & C_{31} \ldots C_{3,n-2} & C_{3,n-1} & 0 \ldots 0 \\ 0 & 0 & 0 & C_{40} \ldots C_{4,n-3} & C_{4,n-2} & 0 \ldots 0 \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot & \cdot & \cdot \quad \cdot \\ 0 & 0 & 0 & 0 \ldots 0 & 0 & 0 \ldots C_{2n,0} \end{vmatrix} . \qquad (4)$$

The even rows furnish the coefficients of the successive Sturm remainders, the ith *remainder* f_{i+1} *being either* $C_{2i,0}\, x^{n-i} + C_{2i,1}\, x^{n-i-1} + \cdots + C_{2i,n-i}$ *or the same taken with the opposite sign, according as there is an even or an odd number of negative constituents in the principal diagonal above the constituent* $C_{2i,0}$.

The advantage of this mode of computing a Sturm's series over the usual long-division method can perhaps be best seen by the reader, if he makes the computation for some equation by both methods. Take, for instance, the equation*

$$x^6 + x^5 - x^4 - x^3 + x^2 - x + 1 = 0,$$

and place $f_1(x)$ equal to the derivative of the left hand member. The two functions f_1 and f_2 are in consequence identical, and f_3 becomes the first remainder. Omitting the first row and column of (1), we have as our fundamental determinant

$$\begin{vmatrix} 6 & 5 & -4 & -3 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 5 & -4 & -3 & 2 & -1 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 6 & 5 & -4 & -3 & 2 & -1 \end{vmatrix},$$

* This equation is found in Serret's *Algebra*, Part I, chap. 6. Owing to an error committed in computing the first remainder, the series of Sturm's functions there given is incorrect. The error does not however affect the conclusion that the roots of the equation are all imaginary.

which reduced to its final form becomes

$$\begin{vmatrix} 6 & 5 & -4 & -3 & 2 & -1 & & & & & \\ & 1 & -2 & -3 & 4 & -5 & 6 & & & & \\ & & 17 & 14 & -27 & 32 & -37 & & & & \\ & & & -8 & -4 & 6 & -8 & 17 & & & \\ & & & & 44 & -114 & 120 & -7 & & & \\ & & & & & -16 & 18 & -6 & 11 & & \\ & & & & & & -86 & 138 & 31 & & \\ & & & & & & & -30 & -46 & 43 & \\ & & & & & & & & 506 & -173 & \\ & & & & & & & & & -331 & 253 \\ & & & & & & & & & & 1 \end{vmatrix}.$$

According to the usual method the third Sturm remainder $-86\,x^2+138x+31$ is obtained by dividing $17x^4+14x^3-27x^2+32x-37$ by $-44x^3+114x^2-120x+7$; fractional coefficients being avoided by multiplying the minuends by appropriate positive numbers. Several of the coefficients occurring in this division exceed 100,000, the largest being 306,480, and after the division the factor 1734 must be removed from the remainder to reduce it to the above form. On the other hand, if the determinant be made the basis of the computation, to obtain the third remainder after the second has been found we have to reduce the matrix

$$\begin{vmatrix} 44 & -144 & 120 & -7 & 0 \\ -8 & -4 & 6 & -8 & 17 \\ 0 & 44 & -114 & 120 & -7 \end{vmatrix}$$

in the manner already described. The largest number occurring in this reduction is 480 and the divisors removed from the second and third rows are 17 and 3. In the computation of the next remainder the ratio of the largest numbers involved in the two methods is approximately 60 : 1, and so on.

6. It will be noticed that the fundamental determinant (1) has the same general form whether read downwards and to the right from the upper left hand corner or read upwards and to the left from the lower right hand corner. If also, as in the example just given, $f_1(x)$ is a polynomial of the n–1th degree so that the first row and column may be erased, the determinant will be bordered both at top and bottom with the coefficients of the same polynomial. A second series of Sturm's functions can then be obtained by working upwards from the lowermost row and reducing all the constituents above the principal diagonal

to zero. The series of functions thus obtained* is identical with the series of remainders which Sturm derived by so conducting the division of $f(x)$ by $f_1(x)$ and each succeeding division thereafter, as to remove the two terms of lowest degree instead of the two terms of highest degree. After each division the factor $-x^2$ is to be removed from the remainder, which is then used as the next divisor.

7. Thus far we have not brought into consideration the functions

$$g_{i+1}(x) = C_{2i+1,0}\,x^{n-i} + C_{2i+1,1}\,x^{n-i-1} + \cdot\cdot\cdot\cdot$$

which are formed from the odd rows of (4) in the same manner as the functions $f_i(x)$ from the even rows. Between the functions f and g there exist relations having, in general,† the form

$$\begin{aligned}
d_0 f_2 &= -a_0 f_1 + b_0 f, \\
d_1 g_2 &= l_1 x f_2 + m_1 f, \\
&\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
d_{2i} f_{i+2} &= l_{2i}\, g_{i+1} + m_{2i} f_{i+1}, \\
d_{2i+1}\, g_{i+2} &= l_{2i+1}\, x f_{i+2} + m_{2i+1}\, g_{i+1}, \\
&\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\
d_{2n-2}\, f_{n+1} &= l_{2n-2}\, x f_n + m_{2n-2}\, g_n;
\end{aligned} \tag{5}$$

in which the l, m, and d denote constants whose values will presently be determined. For consider, say, the $(j+1)$th relation. The functions therein connected are formed from the jth, the $(j+1)$th, and the $(j+2)$th rows. Now immediately before the $(j+2)$th row was reduced to its final form, its significant constituents were the same as those of the jth row but displaced one column towards the right. The first of these constituents, lying to the left of the principal diagonal, was consequently to be reduced to zero. This can be effected by multiplying the row by $|C_{j+1,0}|$ and adding or subtracting the preceding row multiplied by $C_{j,0}$. Hence the three functions are connected in the manner stated. Furthermore if d_j denotes the greatest (positive) common divisor of the constituents of the $(j+2)$th row, which finally is removed to reduce the row to its simplest form, we have $l_j = \pm C_{j,0}$, $m_j = \pm C_{j+1,0}$. To remove the ambiguity of sign which here presents itself, suppose first that $C_{j+1,0}$ is positive. Then in the reduction of the $(j+2)$th row the preceding row, after multiplication with $C_{j,0}$, is to be subtracted. If also $C_{j,0}$ is positive, to obtain the coefficients of the three functions we either take the con-

* This series has been termed a "Sturm'sche Reihe zweiter Gattung." For a detailed discussion see Wendlandt's thesis bearing this title (Göttingen, 1877).

† The investigation following applies only to the general case.

stituents of the three rows without change of sign, or else change the sign of every constituent. It follows that $m_j = C_{j+1,0}$, $l_j = -C_{j,0}$. On the other hand, if $C_{j,0}$ is negative, to obtain the coefficients, we change either the signs of the constituents of the $(j+1)$th and $(j+2)$th rows or the signs of the constituents of the jth row. Hence l_j has the same value as before, but $m_j = -C_{j+1,0}$. The same results will also be obtained by similar reasoning when $C_{j+1,0}$ is negative. Hence l_j is in every case equal to $-C_{j,0}$, while m_j is equal to $+C_{j+1,0}$ or $-C_{j+1,0}$ according as $C_{j+1,0}$ is preceded in the principal diagonal by a positive or a negative constituent.

8. The relations (5) give at once a development of f/f_2 into a continued fraction

$$\lambda_1 + \frac{\mu_1}{\lambda_2 +}\,\frac{\mu_2}{\lambda_3 +}\cdots + \frac{\mu_{2n-1}}{1}, \quad \text{in which}$$

$$\lambda_{2i} = -\frac{l_{2i}}{m_{2i}},\ \lambda_{2i+1} = -\frac{l_{2i+1}x}{m_{2i+1}},\ \mu_i = \frac{d_i}{m_i},\ \cdots\ \mu_{2n-1} = \frac{C_{2n-1,1}}{m_{2n-1}}.$$

From this it follows* that

$$f = \begin{vmatrix} \lambda_1 - \mu_1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & \lambda_2 - \mu_2 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & \lambda_3 - \mu_3 & 0 & \ldots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 \ldots & 1 & \lambda_{2n-1} - \mu_{2n-1} & \\ 0 & 0 & 0 & 0 & 0 \ldots & 0 & 1 & 1 \end{vmatrix}$$

A like expression for f_2 can obviously be obtained by omitting the first row and column and for f_{i+2} by omitting the first $2i+1$ rows and columns.

Consider now the case in which f_1 is of the $n-1$th degree so that we may set $f_2 = f_1$. *The problem of finding for any given numerical value of x the signs of all the successive remainders* — except the last which is a constant — *can evidently then be presented as that of determining the signs of the determinant last given and certain minors of odd order.*† For this purpose a slightly modified form of the determinant will be much more convenient. After the values of the λ and μ above given have been substituted, let each row be cleared of fractions by multiplying it into the modulus of the denominator

* Heine's *Kugelfunctionen*, Bd. I, s. 260–262.

† The use of the partial quotients of a continued fraction of a somewhat different form for the purpose of determining the signs of the remainders was suggested by Sylvester, *Philosophical Magazine*, 1853.

of its constituents. The composition of the determinant will then be the same as would result by constructing it directly from (4) according to the following rules:

(1). The principal diagonal is first to be filled with the same constituents as the principal diagonal of (4), the last constituent being replaced by unity, and every other constituent, beginning with the first, is to be multiplied by x.

(2). The parallel file just below the principal diagonal is to be filled with the constituents of the principal diagonal of (4), beginning with the third, but the sign of each constituent is to be reversed in case it is preceded in the principal diagonal of (4) by a negative constituent. The last constituent of the file is to be placed equal to unity.

(3). The parallel file just above the principal diagonal is to be filled with the negatives of the divisors d used in the reduction of the fundamental determinant. The last constituent of the file is to be placed equal to $-C_{2n-1,1}$.

(4). All other constituents of the determinant are zero.

(5). Finally, if any constituent of the file below the principal diagonal is negative, the signs of the three significant constituents of the row in which it is contained are to be reversed.

Thus, for example, if $f(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$, the fundamental determinant and the divisors d used in its formation are as follows:

$$
\begin{array}{r|rrrrrrrrrr|}
d & & & & & & & & & & \\
\hline
 & 1 & 1 & -4 & -3 & 3 & 1 & & & & \\
 & & 5 & 4 & -12 & -6 & 3 & & & & \\
1 & & & 1 & -8 & -9 & 12 & 5 & & & \\
11 & & & & 4 & 3 & -6 & -2 & & & \\
5 & & & & & -7 & -6 & 10 & 4 & & \\
1 & & & & & & -3 & -2 & 2 & & \\
4 & & & & & & & -1 & 4 & 3 & \\
7 & & & & & & & & -2 & -1 & \\
3 & & & & & & & & & 3 & 2 \\
1 & & & & & & & & & & 1
\end{array},
$$

and the auxiliary determinant formed with the aid of the above rules is

$$\begin{vmatrix}
x & -1 & & & & & & & & \\
1 & 5 & -11 & & & & & & & \\
 & 4 & x & -5 & & & & & & \\
 & & 7 & -4 & 1 & & & & & \\
 & & & 3 & -7x & -4 & & & & \\
 & & & & 1 & -3 & -7 & & & \\
 & & & & & 2 & -x & -3 & & \\
 & & & & & & 3 & 2 & 1 & \\
 & & & & & & & 1 & 3x & -2 \\
 & & & & & & & & 1 & 1
\end{vmatrix}.$$

The calculation of the minors whose signs are desired for the given value of x can be easily made by computing the minors common to the last i rows and columns, i taking successively the values 2, 3, 4, ... Each of these minors can be quickly computed since it is a linear combination of the two preceding, in which the coefficients depend only upon the three new constituents appearing in the minor. This method of determining the signs of Sturm's functions requires the substitution of the given value of x in a single determinant instead of in n functions, as by the usual method. In many cases, at least, the method can be employed with decided advantage.

MIDDLETOWN, CONN., MAY, 1899.